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Thermofield Dynamics of the Heterotic String

— Global Phase Structure —

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ABSTRACT

Physical aspects of the thermofield dynamics of the $D = 10$ heterotic thermal string theory are exemplified through the infrared behaviour of the one-loop dual symmetric cosmological constant in association with the global phase structure of the thermal string ensemble.

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Elaboration of thermal string theories based upon the thermofield dynamics (TFD) [1] has gradually turned out to be a good practical subject of high energy physics [2] - [9]. In a previous paper of ourselves [8], we have succeeded in shedding some light upon the global phase structure of the thermal string excitation in proper reference to the thermal duality relation [10], [11] for the $D = 26$ closed bosonic thermal string theory within the TFD framework. In the present communication, physical aspects of the $D = 10$ heterotic thermal string theory based upon the TFD algorithm are exemplified through the infrared behaviour of the one-loop cosmological constant in respect of the thermal duality symmetry. The global phase structure of the TFD thermal string amplitude is then examined *à la* O'Brien and Tan [10] on the basis of the thermal stability of modular invariance.

Let us start with describing the one-loop cosmological constant $\Lambda(\beta)$ at any finite temperature $\beta^{-1} = kT$ as

$$\Lambda(\beta) = \frac{\alpha'}{2} \lim_{\mu^2 \to 0} \mathrm{Tr} \left[\int_{\infty}^{\mu^2} dm^2 \left(\Delta_B^\beta(p, P; m^2) + \Delta_F^\beta(p, P; m^2) \right) \right] \tag{1}$$

for the $D = 10$ heterotic thermal string theory in the TFD framework, where the string tension σ is expressed in terms of the slope parameter α' as $\sigma = 1/2\pi\alpha'$, p^μ reads loop momentum and P^I lie on the even self-dual root lattice $L = \Gamma_8 \times \Gamma_8$ for the exceptional group $G = E_8 \times E_8$ [12], respectively. Here the thermal propagator $\Delta_{B[F]}^\beta(p, P; m^2)$ of the free closed bosonic [fermionic] string is expressed at $D = 10$ as

$$\begin{aligned} \Delta_{B[F]}^\beta(p, P; m^2) &= \int_{-\pi}^{\pi} \frac{d\phi}{4\pi}\, \mathrm{e}^{i\phi\left(N - \alpha - \bar{N} + \bar{\alpha} - 1/2 \cdot \sum_{I=1}^{16} (P^I)^2\right)} \\ &\times \left(\left[\underset{[\,-\,]}{+} \int_0^1 dx + \frac{1}{2} \sum_{n=0}^{\infty} \frac{\delta[\alpha'/2 \cdot p^2 + \alpha'/2 \cdot m^2 + 2(n - \alpha)]}{\mathrm{e}^{\beta|p_0|} \underset{[\,+\,]}{-} 1} \oint_c dx \right] \right. \\ &\left. \times x^{\alpha'/2 \cdot p^2 + N - \alpha + \bar{N} - \bar{\alpha} + 1/2 \cdot \sum_{I=1}^{16} (P^I)^2 + \alpha'/2 \cdot m^2 - 1} \right) \quad , \end{aligned} \tag{2}$$

where N $[\bar{N}]$ denotes the number operator of the right- [left-] moving mode, the intercept parameter α $[\bar{\alpha}]$ of the right- [left-] mover is eventually fixed at $\alpha = 0$ $[\bar{\alpha} = 1]$ and the contour c is taken as the unit circle around the origin. The $D = 10$ thermal cosmological constant $\Lambda(\beta)$ is immediately reduced to

$$\begin{aligned}
\Lambda(\beta) \;\; = \;\; & -\frac{i\alpha'}{4}\lim_{\mu^2\to 0}\int_{\infty}^{\mu^2} dm^2 \int_{-\infty}^{\infty}\frac{d^Dp}{(2\pi)^D}\sum_{n=0}^{\infty}\delta\left[\frac{\alpha'}{2}p^2+\frac{\alpha'}{2}m^2+2(n-\alpha)\right] \\
& \times\left(\frac{1}{\mathrm{e}^{\beta|p_0|}-1}+\frac{1}{\mathrm{e}^{\beta|p_0|}+1}\right)\sum_{P^I\in L}\int_{-\pi}^{\pi}\frac{d\phi}{4\pi}\mathrm{e}^{-i\phi\left(\alpha-\bar{\alpha}+1/2\cdot\sum_{I=1}^{16}(P^I)^2\right)} \\
& \times\oint_c dx x^{\alpha'/2\cdot p^2-\alpha-\bar{\alpha}+1/2\cdot\sum_{I=1}^{16}(P^I)^2+\alpha'/2\cdot m^2-1} \\
& \times\mathrm{tr}\left[\mathrm{e}^{i\phi(N-\bar{N})}x^{N+\bar{N}}\right] \quad . \qquad (3)
\end{aligned}$$

Let us turn our attention to explicit calculation of the $D = 10$ thermal amplitude $\Lambda(\beta)$. We are then eventually led to the modular parameter integral representation of $\Lambda(\beta)$ at $D = 10$ as follows [10]:

$$\Lambda(\beta) = -8(2\pi\alpha')^{-D/2}\int_E\frac{d^2\tau}{2\pi\tau_2^2}\,K_h(\bar{\tau},\tau;D)\sum_{\ell\in Z;\mathrm{odd}}\exp\left[-\frac{\beta^2}{4\pi\alpha'\tau_2}\,\ell^2\right] \quad , \qquad (4)$$

where

$$\begin{aligned}
K_h(\bar{\tau},\tau;D) \;\; = \;\; & (2\pi\tau_2)^{-(D-2)/2}\,\mathrm{e}^{2\pi i\bar{\tau}}\left[1+480\sum_{m=1}^{\infty}\sigma_7(m)\bar{z}^m\right] \\
& \times\prod_{n=1}^{\infty}(1-\bar{z}^n)^{-D-14}\left(\frac{1+z^n}{1-z^n}\right)^{D-2} \quad , \qquad (5)
\end{aligned}$$

$\overset{[-]}{\tau} = \tau_1 \overset{+}{\underset{[-]}{}} i\tau_2$, $z = x\mathrm{e}^{i\phi} = \mathrm{e}^{2\pi i\tau}$, $\bar{z} = x\mathrm{e}^{-i\phi} = \mathrm{e}^{-2\pi i\bar{\tau}}$, α $[\bar{\alpha}]$ has been fixed at $\alpha = 0$ $[\bar{\alpha} = 1]$, E means the half-strip integration region in the complex τ plane, i.e. $-1/2 \le \tau_1 \le 1/2$; $\tau_2 > 0$, and the full use has been made of an explicit expression of the theta function $\Theta_{\Gamma_8\times\Gamma_8}$ of the root lattice $\Gamma_8 \times \Gamma_8$ [12]. Accordingly, the $D = 10$ thermal amplitude $\beta\Lambda(\beta)$ is identical in every detail with the "E-type" representation of the thermo-partition

function $\Omega_h(\beta)$ of the heterotic string in ref. [10] as required from the equivalence of the thermal cosmological constant and the free energy amplitude. As can readily be envisaged from eqs. (4) and (5), the "E-type" thermal amplitude $\Lambda(\beta)$ is not modular invariant and consequently is annoyed with ultraviolet divergences at the endpoint $\tau_2 \sim 0$ for any value of β.

Let us now postulate the one-loop dual symmetric thermal cosmological constant $\bar{\Lambda}(\beta; D)$ at any space-time dimension D as an integral over the fundamental domain F, i.e. $-1/2 \leq \tau_1 \leq 1/2\,; \tau_2 > 0\,; |\tau| > 1$, of the modular group $SL(2, Z)$ as follows [10] :

$$\bar{\Lambda}(\beta; D) = -\frac{16}{\beta}(2\pi\alpha')^{-D/2} \sum_{(\sigma,\rho)} \int_F \frac{d^2\tau}{2\pi\tau_2^2}\, B(\bar{\tau}, \tau; D) A_{\sigma\rho}(\tau; D) D_{\sigma\rho}(\bar{\tau}, \tau; \beta) \quad , \tag{6}$$

where

$$\begin{aligned} B(\bar{\tau}, \tau; D) &= -\frac{1}{8}(2\pi\tau_2)^{-(D-2)/2}\; \bar{z}^{-(D+14)/24} z^{-(D-2)/24} \\ &\quad \times \left[1 + 480 \sum_{m=1}^{\infty} \sigma_7(m) \bar{z}^m\right] \prod_{n=1}^{\infty} (1 - \bar{z}^n)^{-D-14} (1 - z^n)^{-D+2}, \end{aligned} \tag{7}$$

$$\begin{pmatrix} A_{+-}(\tau; D) \\ A_{-+}(\tau; D) \\ A_{--}(\tau; D) \end{pmatrix} = 8\left(\frac{\pi}{4}\right)^{(D-2)/6} \begin{pmatrix} -[\theta_2(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \\ -[\theta_4(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \\ [\theta_3(0,\tau)/\theta_1'(0,\tau)^{1/3}]^{(D-2)/2} \end{pmatrix} \quad , \tag{8}$$

$$D_{\sigma\rho}(\bar{\tau}, \tau; \beta) = C_\sigma^{(+)}(\bar{\tau}, \tau; \beta) + \rho C_\sigma^{(-)}(\bar{\tau}, \tau; \beta) \quad , \tag{9}$$

$$C_\sigma^{(\gamma)}(\bar{\tau}, \tau; \beta) = (4\pi^2\alpha'\tau_2)^{1/2} \sum_{(p,q)} \exp\left[-\frac{\pi}{2}\left(\frac{\beta^2}{2\pi^2\alpha'}p^2 + \frac{2\pi^2\alpha'}{\beta^2}q^2\right)\tau_2 + i\pi pq\tau_1\right], \tag{10}$$

the signatures σ, ρ and γ read $\sigma, \rho = +,-;\ -,+;\ -,-$ and $\gamma = +,-$, respectively, the summation over p $[q]$ is restricted by $(-1)^p = \sigma$ $[(-1)^q = \gamma]$ and the explicit use has been

made of the Jacobi theta functions $\theta_j(0,\tau);\ j=1,2,3,4$ as well as the Poisson resummation formula. It is almost needless to mention that the $D=10$ thermal amplitude $\beta\bar{\Lambda}(\beta;D=10)$ is literally reduced to the "D-type" representation of the thermo-partition function $\Omega_h(\beta)$ in ref. [10] which in turn guarantees $\bar{\Lambda}(\beta;D=10)=\Lambda(\beta)$ as expected from self-consistency. Let us now examine the algebraic structure of the "D-type" thermal amplitude $\bar{\Lambda}(\beta;D)$ with the arithmetic aid of Appendix B in ref. [10]. Typical theoretical observations are as follows: The scalar product $\sum_{(\sigma,\rho)} A_{\sigma\rho}D_{\sigma\rho}$ is invariant under permutations of the signature, irrespective of the values of β and D, not only for the shifting transformation $\tau\rightarrow\tau+1$ but also for the inversion $\tau\rightarrow-\tau^{-1}$. In addition, $B(\bar{\tau},\tau;D)$ is invariant, irrespective of the value of D, under the action of any modular transformation. Accordingly, the "D-type" thermal amplitude $\bar{\Lambda}(\beta;D)$ is manifestly modular invariant and thus free of ultraviolet divergences for any value of β and D. As a matter of fact, moreover, the generalized duality symmetry [10] $C_\sigma^{(\gamma)}(\bar{\tau},\tau;\beta)=C_\gamma^{(\sigma)}(\bar{\tau},\tau;\tilde{\beta})$ holds for any value of D, where $\tilde{\beta}=2\pi^2\alpha'/\beta$. If and only if $D=10$, on the other hand, the scalar product $\sum_{(\sigma,\rho)} A_{\sigma\rho}D_{\sigma\rho}$ is invariant under the thermal duality transformation $\beta\leftrightarrow\tilde{\beta}$ as a simple and natural consequence of the Jacobi identity $\theta_2^4-\theta_3^4+\theta_4^4=0$ for the theta functions. We are then led to conclude that the thermal duality relation $\beta\bar{\Lambda}(\beta;D)=\tilde{\beta}\bar{\Lambda}(\tilde{\beta};D)$ is manifestly broken for the "D-type" thermal amplitude $\bar{\Lambda}(\beta;D\neq10)$ off the critical dimension.

Let us recall to our remembrance that $\theta_1'^{-1/3}\sim \mathrm{e}^{\pi\tau_2/12};\ \theta_2\sim0;\ \theta_3\sim1;\ \theta_4\sim1$ near $\tau_2\rightarrow\infty$. The infrared behaviour of the "D-type" thermal cosmological constant $\bar{\Lambda}(\beta;D)$ is then asymptotically described at $\tau_2\rightarrow\infty$ as [10]

$$\bar{\Lambda}(\beta;D)=-\frac{16}{\beta}\,(2\pi\alpha')^{-D/2}\int_F\frac{d^2\tau}{2\pi\tau_2^2}\,B(\bar{\tau},\tau;D)\,[A_{-+}(\tau;D)-A_{--}(\tau;D)]\,C_-^{(-)}(\bar{\tau},\tau;\beta)\tag{11}$$

which is in turn paraphrased into the form

$$\bar{\Lambda}(\beta;D)\ =\ -64\sqrt{2}\,(8\pi^2\alpha')^{-D/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\tau_1\exp[i\pi pq\tau_1]\sqrt{\frac{\tilde{\beta}}{\beta}}\int_{\sqrt{1-\tau_1^2}}^{\infty}d\tau_2\ \tau_2^{-(D+1)/2}$$

$$\times \exp\left[-\frac{\pi}{2}\,\tau_2\left(\frac{\beta}{\tilde{\beta}}\,p^2+\frac{\tilde{\beta}}{\beta}\,q^2-\frac{5}{12}(D-10)-6\right)\right] \quad , \tag{12}$$

where $p, q = \pm 1; \pm 3; \pm 5; \cdots$. As can easily be seen from eq. (12), uniform convergence of the "D-type" thermal amplitude $\bar{\Lambda}(\beta; D)$ is assured at any value of β if and only if $D < 2/5$. Of principal concern with us is the case $D = 10$, anyhow. Infrared convergence of the $D = 10$ "D-type" TFD amplitude $\bar{\Lambda}(\beta; D = 10)$ is then guaranteed if and only if either $(2+\sqrt{2})\pi\sqrt{\alpha'} = \beta_H < \beta < \infty$ or $0 < \beta < \tilde{\beta}_H = (2-\sqrt{2})\pi\sqrt{\alpha'}$, where β_H $[\tilde{\beta}_H]$ reads the inverse [dual] Hagedorn temperature of the heterotic thermal string. Explicit calculation of the τ_2 integral in eq. (12) is readily performed for the case $D < 2/5$ and yields

$$\begin{aligned}
\bar{\Lambda}(\beta; D) \;&=\; -\frac{128}{\sqrt{\pi}}(16\pi\alpha')^{-D/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi p q \tau_1] \\
&\times\sqrt{\frac{\tilde{\beta}}{\beta}}\left(\frac{\beta}{\tilde{\beta}}\,p^2+\frac{\tilde{\beta}}{\beta}\,q^2-\frac{5}{12}(D-10)-6\right)^{(D-1)/2} \\
&\times\Gamma\left[-\frac{D-1}{2}, \frac{\pi}{2}\sqrt{1-\tau_1^2}\left(\frac{\beta}{\tilde{\beta}}\,p^2+\frac{\tilde{\beta}}{\beta}\,q^2-\frac{5}{12}(D-10)-6\right)\right] \quad ,
\end{aligned} \tag{13}$$

irrespective of the value of β, where Γ is the incomplete gamma function of the second kind. We are now in the position to carry out the dimensional regularization in the sense of analytic continuation of the TFD amplitude (13). The right-hand side of eq. (13) indeed obeys the thermal duality symmetry $\beta\bar{\Lambda}(\beta; D) = \tilde{\beta}\bar{\Lambda}(\tilde{\beta}; D)$ and brings forth the correct analytic continuation from $D < 2/5$ to higher values of D, i.e. $D = 10$. We can therefore define the dimensionally regularized, $D = 10$ one-loop dual symmetric thermal cosmological constant $\hat{\Lambda}(\beta)$ by

$$\begin{aligned}
\hat{\Lambda}(\beta) \;&=\; -\frac{2}{\beta}(8\pi\alpha')^{-(D-1)/2}\sum_{(p,q)}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp[i\pi p q \tau_1] \\
&\times\left(\frac{\beta^2}{2\pi^2\alpha'}\,p^2+\frac{2\pi^2\alpha'}{\beta^2}\,q^2-6\right)^{(D-1)/2} \\
&\times\Gamma\left[-\frac{D-1}{2}\,,\;\frac{\pi}{2}\sqrt{1-\tau_1^2}\left(\frac{\beta^2}{2\pi^2\alpha'}\,p^2+\frac{2\pi^2\alpha'}{\beta^2}\,q^2-6\right)\right] \;; \quad D = 10
\end{aligned} \tag{14}$$

which manifestly satisfies the thermal duality relation $\beta\hat{\Lambda}(\beta) = \tilde{\beta}\hat{\Lambda}(\tilde{\beta})$. It must be emphasized that the present dimensional regularization based upon the TFD algorithm is in full accordance with the thermal stability of the fundamental properties such as modular invariance.

Let us next examine the singularity structure of the dimensionally regularized, $D = 10$ dual symmetric thermal amplitude $\hat{\Lambda}(\beta)$. The position of the singularity $\beta_{|p|,|q|}$ is determined by solving $\beta/\tilde{\beta}\cdot p^2 + \tilde{\beta}/\beta\cdot q^2 - 6 = 0$ for every allowed (p, q) in eq. (14). We then obtain two sets of solutions with $|pq| \leq 3$ as follows: (i) $\beta_{1,1} = \beta_H = (\sqrt{2}+1)\pi\sqrt{2\alpha'}$; $\tilde{\beta}_{1,1} = \tilde{\beta}_H = (\sqrt{2}-1)\pi\sqrt{2\alpha'}$, (ii) $\beta_{1,3} = \tilde{\beta}_{3,1} = \sqrt{3}\pi\sqrt{2\alpha'}$; $\beta_{3,1} = \tilde{\beta}_{1,3} = 1/\sqrt{3}\cdot\pi\sqrt{2\alpha'}$. In particular, $\beta_{1,1}$ and $\tilde{\beta}_{1,1}$ form the leading branch points of the square root type at β_H and $\tilde{\beta}_H$, respectively. Moverover, β_H^{-1} $[\tilde{\beta}_H^{-1}]$ represents the lowest temperature singularity for the physical β [dual $\tilde{\beta}$] channel. Both $\beta_{1,3}$ and $\beta_{3,1}$ are ordinary points and consequently left out of consideration. It is of practical importance to note that there exists no self-dual leading branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$ as well as any non-leading branch point on the physical sheet of the inverse temperature. The present theoretical observation based upon the TFD free energy amplitude of the $D = 10$ heterotic thermal string yields a striking contrast to the previous argument by ourselves [8] for the $D = 26$ closed bosonic thermal string theory in the TFD framework. We are now in the position to touch upon the global phase structure of the $D = 10$ heterotic thermal string ensemble. Analysis is performed *à la* ref. [10], ref. [13] and ref. [14] through the microcanonical ensemble paradigm outside the analyticity domain of the canonical ensemble. There will then appear three phases in the sense of the thermal duality symmetry as follows [8], [10], [13]: (i) the β channel canonical phase in the domain $(2+\sqrt{2})\pi\sqrt{\alpha'} = \beta_H \leq \beta < \infty$, (ii) the dual $\tilde{\beta}$ channel canonical phase in the domain $0 < \beta \leq \tilde{\beta}_H = (2-\sqrt{2})\pi\sqrt{\alpha'}$ and (iii) the self-dual microcanonical phase in the domain $\tilde{\beta}_H < \beta < \beta_H$. In sharp contrast to the global phase structure of the $D = 26$ closed bosonic thermal string ensemble [8], however, there will occur no effective splitting of the microcanonical region because of

the absence of the self-dual branch point at $\beta_0 = \tilde{\beta}_0 = \pi\sqrt{2\alpha'}$ as well as any secondary singularity. As a consequence of the self-duality of the microcanonical phase, therefore, it may be possible to claim that the so-called maximum temperature of the $D = 10$ heterotic string excitation is asymptotically described at least at the one-loop level as $\tilde{\beta}_H^{-1}$ [β_H^{-1}] in replacement of $\beta_0^{-1} = \tilde{\beta}_0^{-1}$ for the physical β [dual $\tilde{\beta}$] channel. It seems almost needless to mention that the fruitful thermodynamical investigation of string excitations will be prerequisite for the real solid substantiation of the present novel hypothesis on the "true" maximum temperature in proper reference to the global phase structure of the $D = 10$ heterotic thermal string ensemble. It is hoped that we can shed some light upon this unfathomable subject with the new-fashioned aid of the D-brane paradigm [15] in a future communication.

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